PhD thesis on TASEP, Bethe ansatz and KPZ universality
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A strong liking for calculations is very much advised.

Statistical mechanics was born at the end of the 19th century to explain from first principles the properties of matter with a huge number of constituents. For systems *at equilibrium*, which do not evolve in time, it consists in trading the precise trajectories of atoms and molecules with probabilities, thus reducing the evaluation of physical observables to a counting problem.

Systems at equilibrium are however a rarity: most phenomena in nature, including turbulence, chemical reactions and life itself exhibit large scale currents of matter and energy, and are thus fundamentally *out of equilibrium*. These currents typically induce long distance correlations, a property shared with the special class of critical equilibrium systems, which are described mathematically by universal laws independent of scale. A fruitful approach has thus been to extend these notions to non-equilibrium phenomena.

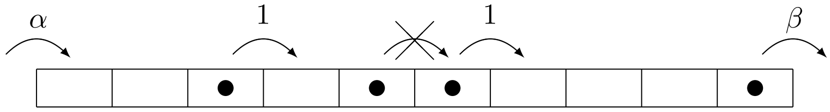

Figure 1: One-dimensional TASEP connected to reservoirs of particles.

A very simple genuinely non-equilibrium model is the one-dimensional totally asymmetric simple exclusion process (TASEP) [1], a lattice model of classical hard core particles moving in the same direction under the action of an external field. The dynamics is that of an irreversible Markov process in continuous time. When the system is connected to reservoirs of particles at both ends, it is defined as follows, see also figure 1: each particle in the bulk of the system can hop of one site to the right with rate 1 if the destination site is empty; a particle can enter the system on the first site, if it is empty, with rate α; a particle on the last site can leave with rate β.

At large scales, when the number of lattice sites and time both go to infinity in a precise way, TASEP is known to belong to a prominent non-equilibrium universality class known as KPZ, which has recently become a focus of convergent efforts between experimental and theoretical physicists and mathematicians [2]. The name comes from an a priori very different subject, the growth of interfaces, for which a continuous model is the Kardar-Parisi-Zhang (KPZ) equation, introduced in 1986. The KPZ equation belongs to a class of very singular non-linear stochastic partial differential equations, whose well-posedness has only been proven recently by Hairer, for which he got the fields medal in 2014.

KPZ universality is a collection of results stating that the large scale fluctuations of some observables (the current of particles in driven systems such as TASEP, the position of growing interfaces between stable and metastable thermodynamic phases, or the free energy for directed polymers in random media) have exactly the same statistics, independently of most microscopic details of the models studied. This can be seen as a kind of extension of the central limit theorem for some classes of highly correlated random variables, leading to non-Gaussian distributions [3]. In some regimes, an exact description of the statistics closely related to random matrix theory exists, but a unified picture is still missing.

Universality tells us that it is possible to obtain results for the whole KPZ class by only studying a single model in it. Furthermore, since there is much freedom in the precise choice of the model, one can as well choose a convenient one. As it turns out, TASEP is convenient, or more precisely, *exactly solvable* [1]. Indeed, TASEP is an integrable model that can be solved using the Bethe ansatz method, which was invented initially to diagonalize exactly the Heisenberg quantum spin chain. Bethe ansatz is in itself a beautiful topic in mathematical physics with applications in condensed matter, high energy physics and combinatorics, and which is often the last resort for treating strongly coupled problems in a non-perturbative way when everything else fails. For TASEP with periodic boundary conditions, it recently lead to exact formulas [4] for the statistics of the current on the scale where the system relaxes to its non-equilibrium steady state.

During the PhD, the student will apply Bethe ansatz methods to KPZ universality. A goal will be to generalize the results of [4] to the case of TASEP in contact with reservoirs of particles, in order to understand precisely the effect of the environment. It will require a combinations of analytical methods (asymptotic calculations similar to those in [5] are expected), computer algebra and numerics.

References

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