Homework 2

Marc Potters

Introduction to Random Matrix Theory
and its applications to data analysis

January 25, 2019
due February 8, 2019

Exercise 1. $\mathrm{Im}\, g_N(x - i\eta)/\pi$ is a good approximation to $\rho(x)$ for small positive η, where $g_N(z)$ is the sample Stieljes transform ($g_N(z) = (1/N)\sum_k 1/(z - \lambda_k)$). Numerically generate a Wigner matrix of size N and $\sigma^2 = 1$.

1. For three values of η $\{1/N, 1/\sqrt{N}, 1\}$, plot $\mathrm{Im}\, g_N(x - i\eta)/\pi$ and the theoretical $\rho(x)$ on the same plot for x between -3 and 3.

2. Compute the error as a function of η where the error is $(\rho(x) - \mathrm{Im} g_N(x - i\eta)/\pi)^2$ summed for all values of x between -3 and 3 spaced by intervals of 0.01. Plot this error for η between $1/N$ and 1. You should see that $1/\sqrt{N}$ is very close to the minimum of this function.

Exercise 2. We saw in class that the Stieljes transform of a large Wishart matrix (with $q = N/T$) should be given by

$$g(z) = \frac{z + q - 1 \pm \sqrt{(z + q - 1)^2 - 4qz}}{2qz} \tag{1}$$

where the sign of the square-root should be chosen such that $g(z) \to 1/z$ when $z \to \pm\infty$.

1. Show that the zeros of the argument of the square-root are given by $\lambda_\pm = (1 \pm \sqrt{q})^2$.

2. The function
$$g(z) = \frac{z + q - 1 - \sqrt{z - \lambda_-}\sqrt{z - \lambda_+}}{2qz} \tag{2}$$
should have the right properties. Show that it behaves as $g(z) \to 1/z$ when $z \to \pm\infty$. By expanding in powers of $1/z$ up to $1/z^3$ compute the first and second moments of the Wishart distribution.

3. Show that Eq. (2) is regular at $z = 0$ when $q < 1$. In that case, compute the first inverse moment of the Wishart matrix $\phi(\mathbf{E}^{-1})$. What happens when $q \to 1$? Show that Eq. (2) has a pole at $z = 0$ when $q > 1$ and compute the value of this pole.

4. The non-zero eigenvalues should be distributed according to the Marčenko-Pastur distribution
$$\rho_q(x) = \frac{\sqrt{(x-\lambda_-)(\lambda_+ - x)}}{2\pi q x}. \tag{3}$$
Show that this distribution is correctly normalised when $q < 1$ but not when $q > 1$. Use what you know about the pole at $z = 0$ in that case to correctly write down $\rho_q(x)$ when $q > 1$.

5. In the case $q = 1$, Eq. (3) has an integrable singularity at $x = 0$. Write a simpler formula for $\rho_1(x)$. Let u be the square of an eigenvalue from a Wigner matrix of unit variance, i.e. $u = y^2$ where y is distributed according to the semi-circular law $\rho(y) = \sqrt{4 - y^2}/(2\pi)$. Show that u is distributed according to $\rho_1(x)$. This result is *a priori* not obvious as a Wigner matrix is symmetric while the square matrix $\mathbf{H}$ is generally not, nevertheless moments of high dimensional matrices of the form $\mathbf{H}\mathbf{H}^\intercal$ are the same whether the matrix $\mathbf{H}$ is symmetric or not.

6. Generate three matrices $\mathbf{E} = \mathbf{H}\mathbf{H}^\intercal/T$ where the matrix $\mathbf{H}$ is a $N \times T$ matrix of iid Gaussian numbers of variance 1. Choose a large N and three values of T such that $q = N/T$ equals $\{1/2, 1, 2\}$. Plot a normalised histogram of the eigenvalues in the three cases vs the corresponding Marčenko-Pastur distribution, don't show the peak at zero. In the case $q = 2$, how many zero eigenvalues do you expect? How many do you get?