Final exam: Random Matrix Theory and Applications

Marc Potters

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1 Potentially useful formulae

1.1 Basic Transforms

$$\tau(\mathbf{A}) = \frac{1}{N}\mathbb{E}\,\mathrm{Tr}\,\mathbf{A} \tag{1}$$

$$\mathbf{G}_{\mathbf{A}}(z) = (z\mathbf{1} - \mathbf{A})^{-1}, \tag{2}$$

$$g_N^{\mathbf{A}}(z) = \frac{1}{N}\,\mathrm{Tr}\,(\mathbf{G}_{\mathbf{A}}(z)) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{z-\lambda_k}, \tag{3}$$

$$\mathfrak{g}(z) = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\tau(\mathbf{A}^k) \tag{4}$$

$$t_A(\zeta) = \zeta\mathfrak{g}_A(\zeta) - 1 \tag{5}$$

$$= \sum_{k=1}^{\infty}\frac{\tau(\mathbf{A}^k)}{\zeta^k}. \tag{6}$$

$$\lim_{\eta\to 0+}\mathrm{Im}\,\mathfrak{g}(x - \mathrm{i}\eta) = \pi\rho(x). \tag{7}$$

$$\lim_{\eta\to 0+}\mathrm{Im}\,t(x - \mathrm{i}\eta) = \pi x\rho(x). \tag{8}$$

1.2 Free Transforms

$$R(g) = z(g) - \frac{1}{g}. \tag{9}$$

where $z(g)$ is the functional inverse of $\mathfrak{g}(z)$.

$$S_A(t) = \frac{t+1}{t\zeta_A(t)}, \tag{10}$$

where $\zeta_A(t)$ is the functional inverse of $t_A(\zeta)$ given by Eq. (5).

$$S_A(t) = \frac{1}{R_A(tS_A(t))}, \quad R_A(g) = \frac{1}{S_A(gR_A(g))}. \tag{11}$$

1.3 Identity Matrix

$$\mathfrak{g}_{\mathbf{1}}(z) = \frac{1}{1-z}, \qquad t_{\mathbf{1}}(\zeta) = \frac{1}{1-\zeta},$$
$$R_{\mathbf{1}}(x) = 1, \qquad S_{\mathbf{1}}(t) = 1.$$

1.4 Wigner matrix

For a Wigner Matrix the large N limit Stieltjes transform satisfies

$$\frac{1}{\mathfrak{g}(z)} = z - \sigma^2 \mathfrak{g}(z). \tag{12}$$

whose solution is

$$\mathfrak{g}(z) = \frac{z - z\sqrt{1 - 4\sigma^2/z^2}}{2\sigma^2}. \tag{13}$$

$$R_{\mathbf{X}}(x) = \sigma^2 x. \tag{14}$$

1.5 Wishart matrices

A sample covariance matrix is written as

$$\mathbf{E} = \frac{1}{T}\mathbf{H}\mathbf{H}^T \tag{15}$$

For a white Wishart ($\mathbf{H}$ has unit variance IID elements) the law of the elements is

$$P(\mathbf{W}) \propto \exp\left[-\frac{N}{2}\operatorname{Tr} V(\mathbf{W})\right], \tag{16}$$

where

$$V(\mathbf{W}) = (1 - q^{-1}) \log \mathbf{W} + q^{-1}\mathbf{W}. \tag{17}$$

The limiting Stieltjes transform of the white Wishart satisfies

$$\frac{1}{\mathfrak{g}(z)} = z - 1 + q - qz\mathfrak{g}(z). \tag{18}$$

whose solution is

$$\mathfrak{g}(z) = \frac{z - (1 - q) - z\sqrt{(1 - \lambda_+/z)(1 - \lambda_-/z)}}{2qz}. \tag{19}$$

$$\lambda_\pm = (1 \pm \sqrt{q})^2.$$

$$R_{\mathbf{W}}(x) = \frac{1}{1 - qx} \quad S_{\mathbf{W}}(t) = \frac{1}{1 + qt}. \tag{20}$$

For a non-white Wishart we have

$$\mathbf{E}_{\mathbf{C}} = \mathbf{C}^{1/2}\mathbf{W}\mathbf{C}^{1/2}, \tag{21}$$

A sample covariance matrix with both spatial ($\mathbf{C}$) and temporal ($\mathbf{K}$) covariance has

$$\mathbf{E} = \mathbf{C}^{1/2}\mathbf{H}\mathbf{K}\mathbf{H}^T\mathbf{C}^{1/2}, \tag{22}$$

with $\mathbf{H}$ a rectangular matrix with IID elements. Its S-transform is given by

$$S_{\mathbf{E}}(t) = \frac{S_{\mathbf{C}}(t)S_{\mathbf{K}}(qt)}{1 + qt} \tag{23}$$

1.6 Beta and Orthogonal ensemble

The law of the elements in the beta-ensemble ($\beta = 1$ is the orthogonal ensemble)

$$P(\{\mathbf{M}\}) \propto \exp\left\{-\frac{\beta N}{2}\operatorname{Tr} V(\mathbf{M})\right\}, \tag{24}$$

gives the following law for the eigenvalues

$$P(\{\lambda_i\}) = Z_N^{-1}\exp\left\{-\frac{\beta}{2}\left[\sum_{i=1}^{N} NV(\lambda_i) - \sum_{\substack{i,j=1 \\ j\neq i}}^{N}\log|\lambda_i - \lambda_j|\right]\right\}. \tag{25}$$

1.7 Stochastic Calculus and DBM

For a stochastic process

$$\mathrm{d}X_t = \mu(X(t), t)\mathrm{d}t + \sigma(X(t), t)\mathrm{d}B_t, \tag{26}$$

we have Ito's lemma

$$\mathrm{d}F_t = \frac{\partial F}{\partial X}\mathrm{d}X_t + \left[\frac{\partial F}{\partial t} + \frac{\sigma^2(X(t), t)}{2}\frac{\partial^2 F}{\partial^2 X}\right]\mathrm{d}t, \tag{27}$$

Dyson Brownian motion

$$\mathrm{d}\lambda^i = \sqrt{\frac{2}{N}}\mathrm{d}B_i + \frac{1}{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{\mathrm{d}t}{\lambda^i - \lambda^j}, \tag{28}$$

Under BDM the limiting Stieltjes transform satisfies Burgers' equation

$$\frac{\partial \mathfrak{g}}{\partial t} = -\mathfrak{g}\frac{\partial \mathfrak{g}}{\partial z}. \tag{29}$$

whose solution is

$$\mathfrak{g}_t(z) = \mathfrak{g}_0(z - \mathfrak{g}_t(z)).$$

1.8 HCIZ Integral

$$I(\mathbf{A},\mathbf{B}) := \left\langle \exp\left(\frac{N}{2}\operatorname{Tr}\mathbf{AOBO}^T\right)\right\rangle_{\mathbf{O}}, \tag{30}$$

If $\mathbf{A}$ is rank-1 ($\mathbf{A} = a\mathbf{v}\mathbf{v}^T$) we define

$$H_{\mathbf{B}}(a) = \lim_{N\to\infty}\frac{2}{N}\log\left\langle \exp\left(\frac{N}{2}\operatorname{Tr}\mathbf{AOBO}^T\right)\right\rangle_{\mathbf{O}}. \tag{31}$$

For a not too large we have

$$H_{\mathbf{B}}(a) = \int_0^a R_{\mathbf{B}}(x)dx. \tag{32}$$

with $R_{\mathbf{B}}(a)$ the R-transform defined by (9).

1.9 Freeness

In free probabilities, given two random variables A, B. We say they are **free** if for any polynomials $p_1,\dots,p_n$ and $q_1,\dots,q_n$ such that

$$\tau(p_k(A)) = 0, \quad \tau(q_k(B)) = 0, \quad \forall k,$$

we have

$$\tau\left(p_1(A)q_1(B)p_2(A)q_2(B)\cdots p_n(A)q_n(B)\right) = 0.$$

The R-transform is related to the free cumulants by

$$R_A(g) = \sum_{k=1}^{\infty}\kappa_k(A)g^{k-1}. \tag{33}$$

For the S-tranform of a unit trace variable ($\tau(A) = 1$)

$$S_A(t) = 1 - \kappa_2(A)t + O(t^2) \tag{34}$$

Subordination relation under free addition and free product

$$\mathfrak{g}_{A+B}(z) = \mathfrak{g}_A\left(z - R_B(\mathfrak{g}_{A+B}(z))\right), \tag{35}$$

$$t_{AB}(\zeta) = t_A\left(\zeta S_B(t_{AB}(\zeta))\right), \tag{36}$$

1.10 Outliers

Under an additive rank-1 perturbation of size a, we have

$$\lambda_1 = R(1/a) + a \text{ for } a > 1/g_+. \tag{37}$$

with

$$g_+ = \sup_{z>\lambda_+} \mathfrak{g}(z) = \mathfrak{g}(\lambda_+).$$

The square eigenvector overlap is given by

$$|\mathbf{u}_1^T \mathbf{v}|^2 = 1 - a^{-2} R'(a^{-1}).$$

1.11 RIE estimator

When $\mathbf{E}$ is a noisy version of $\mathbf{C}$, the optimal rotationally invariant estimator of $\mathbf{C}$ is given by

$$\Xi(\mathbf{E}) = \sum_k \xi_k \mathbf{v}_k \mathbf{v}_k^T, \quad \text{with} \quad \mathbf{E} = \sum_k \lambda_k \mathbf{v}_k \mathbf{v}_k^T. \tag{38}$$

$$\xi(\lambda) = \frac{\lim_{\eta \to 0^+} \operatorname{Im} H(\lambda - i\eta)}{\lim_{\eta \to 0^+} \operatorname{Im} \mathfrak{g}(\lambda - i\eta)}, \tag{39}$$

where we denote

$$H(z) := \tau(\mathbf{C}G(z)). \tag{40}$$

In the additive case

$$\xi(\lambda) = \lambda - \frac{\lim_{\eta \to 0^+} \operatorname{Im} R_{\mathbf{B}}(g_{\mathbf{E}}(z)) g_{\mathbf{E}}(z)}{\lim_{\eta \to 0^+} \operatorname{Im} g_{\mathbf{E}}(z)}, \quad z = \lambda - i\eta, \tag{41}$$

where $R_{\mathbf{B}}(g)$ is the R-transform of the additive noise.

In the multiplicative case

$$\xi(\lambda) = \lambda \frac{\lim_{\eta \to 0^+} \operatorname{Im} S_{\mathbf{B}}(t_{\mathbf{E}}(z)) t_{\mathbf{E}}(z)}{\lim_{\eta \to 0^+} \operatorname{Im} t_{\mathbf{E}}(z)}, \quad z = \lambda - i\eta, \tag{42}$$

where $S_{\mathbf{B}}(t)$ is the S-transform of the multiplicative noise.

When the multiplicative noise is a white Wishart with parameter q,

$$\xi(\lambda) = \frac{\lambda}{|1 + q t_{\mathbf{E}}(\lambda)|^2}. \tag{43}$$