Intersnhip / Thesis topic:
Quantum walks and path integrals

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Discrete-time quantum walks (DQWs) can be used as discretizations of the Dirac equation, which describes the motion of relativistic quantum particles of matter. Such discretizations are both unitary and strictly local, as the Dirac equation itself. In practice, DQWs can be implemented with quantum technologies and hence be used as actual quantum simulators (i.e., specific-task quantum computers, analog or digital), in particular for the simulation of fundamental-physics dynamics.

The path integral offers an alternative, insightful method to quantize a given classical theory (the usual quantization procedure is canonical quantization); this method is grounded on the principle of least action. As such, the path integral is particularly useful in semi-classical treatments. Moreover, the generalization of the path integral to a very high number of degrees of freedom, namely, the functional integral, which quantizes fields rather than only particles, has been truly fundamental in the development of quantum field theory. The link between DQWs and the path integral is as old as both topics (1940's).

The aim of the present thesis is twofold. In the one-particle sector, i.e., regarding path (rather than functional) integrals, the aim is to define a path integral for the Dirac equation based on a DQW, with an associated one-particle Dirac Lagrangian; there will be difficulties coming from (i) the particularities of the DQW discretization, but also from (ii) the Dirac equation itself. Preliminar works on this topic include Refs. [1, 2]. In the multiparticle sector, i.e., at the level of fields, the aim is to construct a fermionic functional integral based on DQWs, which up to our knowledge has never been done. Several preliminary works will be of useful guidance for this second goal [3–7].

I. EXTENDED DESCRIPTION OF THE TOPIC

A. A fundamental-physics-oriented introduction to discrete-time quantum walks

Quantum walks (QWs) are models of quantum transport on a spacetime lattice. There are two classes of QWs: discrete-time (DQWs), and continuous-time ones. We focus on DQWs. The two fundamental properties of DQWs are the following: (i) unitarity, i.e., "quantumness" – which is not specific to discrete-time QWs but is also present in continuous-time ones –, and (ii) strict locality, which is specific to discrete-time QWs and corresponds to the fact that these DQWs are (quantum) automata, that is, the state of the walker at time $t + \epsilon$ and position x only depends on the states which at time t are within a bounded spatial neighborhood around x, and the size of this neighborhood is fixed.

This "strict locality" property could also be called "relativistic locality", in the sense that there is an upper bound on the speed at which information can propagate in the model. Actually, the relationship between DQWs and relativistic physics is truly fundamental: DQWs can be used to discretize the Dirac equation, which describes the motion of relativistic particles of matter.

Finally, DQWs can be implemented with quantum technologies [8], i.e., they can serve as tools for quantum simulation, that is, simulating a quantum system with another quantum system in order to make the simulation tractable (no exponential growth of the number of degrees of freedom).

B. On the interest of path integrals in physics

The modern history of path integrals starts with Feynman in the 1940's and 1950's [9, 10]. The path-integral method is a quantization method, alternate to canonical quantization, and which is the quantum version of the principle of least action. The first interest of path integrals is thus conceptual, as a principle-of-least-action perspective on any quantum mechanical problem. On the practical side, path integrals have found particularly relevant applications in certain quantum mechanical problems [11]. In particular, path integrals are often extremely useful in semi-classical treatments [12].

But, despite the previous benefits outlined, path integration owes its importance in modern physics to its generalization to a system of a high number of degrees of freedom, such as quantum field theory [12]; this generalization is called functional integration[1], and it achieves the goal of quantizing fields rather than only particles. In particular, the quantization of non-Abelian gauge theories by Faddeev and Popov would have been almost impossible with the canonical approach [12]. Moreover, the path-integral method has highlighted deep mathematical

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[1] Although we sometimes keep using the denomination "path integration" and let the context disambiguate between path and functional integration.

relationships between quantum field theory and statistical mechanics of phase transitions; these relationships would have been difficult to perceive otherwise [12, 13]. Finally, notice that path-integral quantization is manifestly Lorentz covariant, while canonical quantization is not [14].

C. Dirac path integrals with discrete-time quantum walks

The relationship between path integrals and DQWs is as old as both topics. Indeed, the first DQW-like discretization of the Dirac equation appears in Feynman's work in the 1940's [15], and in his book on path integrals [11]. In Problem 2.6 of that book, it is suggested to write down the propagator of the Dirac equation thanks to a certain propagation model in discrete time *and space*[2], and this propagation model is, up to unitarity, a DQW (although this name was coined only much later, with Aharonov's paper [16]). Since then, other works have suggested writing the propagator of the Dirac equation with a true DQW [1, 2].

1. *One-particle path integral*

Up to our knowledge, in most works in which the propagator of the Dirac equation is written thanks to a DQW, no Lagrangian is defined, and so the path-integral expression cannot be written under the insightful usual form. We would like to see what are our options to define a one-particle Lagrangian from a DQW, that would enter the expression of the one-particle path integral.

One first issue is related to the particularities of the DQW discretization of the Dirac equation: this discretization is characterized by a certain one-time-step evolution operator (OEO), from which it is not clear how to define a one-particle Lagrangian. The way to go may probably be to use the effective Hamiltonian of the DQW, defined as the log of the OEO. That being said, there exists another Hamiltonian that one can construct from the OEO, namely, the local Hamiltonian of the DQW [17].

A second issue is related to the Dirac equation itself. Indeed, defining a one-particle Dirac Lagrangian from the Dirac Hamiltonian seems not to work, i.e., the standard procedure is ill-defined in that case. Finally, there is a last issue also related to the Dirac equation itself. Suppose we manage, despite the previous difficulty, to define a one-particle Dirac Lagrangian from the Dirac Hamiltonian. Since the Hamiltonian of the Dirac equation is matrix valued, naively we may expect that the Lagrangian will also be matrix valued. But the Lagrangian is usually always a scalar, so how to interpret a matrix-valued Lagrangian? A way to circumvent this problem may be to always work with matrix components instead of the full matrix at once.

2. *Functional integral*

Up to our knowledge, most works (except Ref. [6], see below) relating path integrals to DQWs deal only with the one-particle sector, i.e., the integral is indeed a path integral and not a functional integral. We would like to define a fermionic functional integral, i.e., a functional integral for Dirac *fields*, based on DQWs.

There are already two works defining an action for Dirac fermions based on DQWs: the first defines a one-time-step action [4], and the second a two-time-steps action, as in usual lattice gauge theory [7]. The first idea would be to define a functional integral with one of the two previous actions. In the above-mentioned works [4, 7], several results of classical field theory are provided (e.g., a Noether's theorem), which will be of useful guidance. A generalization of Ref. [4] to curved spacetimes is given in Ref. [5], and so such generalizations of the functional integral may be envisaged.

Finally, let us mention that in Ref. [6] a functional integral in discrete spacetime and based on quantum automata is provided for bosons interacting via a ϕ^4 term, but fermions are not treated. Fermions could be treated in a similar way based on Ref. [3], which is also a way we should explore since it would also lead to a fermionic functional integral based on DQWs.

II. SUPERVISORS

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[2] Notice that the standard expression of the path integral is also, before going to the continuum, discrete in time, but never discrete in space.

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